Generative Modeling with Optimal Transport Maps

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Overview

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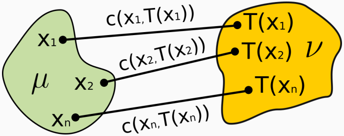
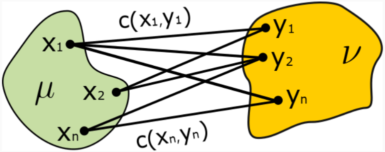
Introduction to Optimal Transport



(a) Monge formulation (1785).



(b) Kantorovich formulation (1942).

Monge Problem (MP):

- Cost$(\mu, \nu) \overset{\text{def}}{=} \inf_{T_{\#}\mu=\nu} \int_{\mathcal{X}} c\left(x, T(x)\right) d\mu(x).$
- The feasible set of solutions can be empty.
- MP does not allow mass splitting.

Kantorovich Problem (KP):

- Cost$(\mu, \nu) \overset{\text{def}}{=} \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c\left(x, y\right) d\pi(x, y).$

Kantorovich Dual Problem (Kantorovich, 1942, KDP) :

$$\text{Cost}(\mu,\nu) = \sup_{(u,v)} \left\{ \int_{\mathcal{X}} u(x) d\mu(x) + \int_{\mathcal{Y}} v(y) d\nu(y) \colon u(x) + v(y) \leq c(x,y) \right\},$$

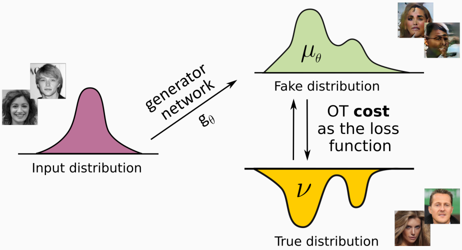
Using c-transform, i.e., $v^c(x) = \inf_{y \in \mathcal{Y}} \{c(x,y) - v(y)\}$, KDP becomes (Villani, 2008, §5):

$$\text{Cost}(\mu,\nu) = \sup_{v} \left\{ \int_{\mathcal{X}} v^c(x) d\mu(x) + \int_{\mathcal{Y}} v(y) d\nu(y) \right\}$$

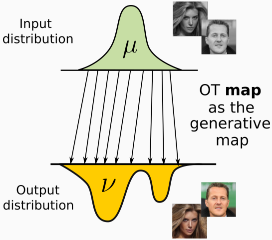
- KDP allows optimization over one functional u or v.

Overview

Optimal Transport in Generative Models



(a) OT cost as the loss for the generator.



(b) OT map as the generative model.

Optimal Transport as the Generative Map

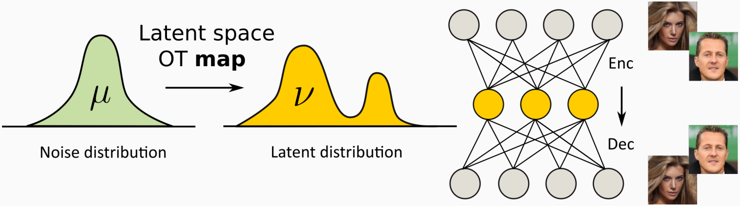


Figure 3: The pipeline of most prevalent approaches (Taghvaei & Jalali, 2019; Makkuva et al., 2020; Korotin et al., 2021).

- Compute OT maps in latent spaces of autoencoders.
- OT maps are not considered in ambient spaces.
- Lack scalability due to poor expressivity of ICNNs.

Overview

Equal Dimensions of Input and Output

With $\psi(y) \stackrel{\text{def}}{=} \frac{1}{2}\|y\|^2 - v(y)$, the c-transform of Kantorovich potential $v(y)$ becomes:

$$\begin{aligned} v^c(x) &= \inf_{y \in \mathbb{R}^D} \left\{ \frac{1}{2}\|x - y\|^2 - v(y) \right\} \\ &= \frac{1}{2}\|x\|^2 - \sup_{y \in \mathbb{R}^D} \{\langle x, y \rangle - \psi(y)\} = \frac{1}{2}\|x\|^2 - \overline{\psi}(x). \end{aligned} \tag{1}$$

Substituting (1), KDP simplifies to

$$\text{Constant}(\mu, \nu) - \inf_{\psi} \left\{ \sup_{T} \int_{\mathcal{X}} \{\langle x, T(x) \rangle - \psi\big(T(x)\big)\} \, d\mu(x) + \int_{\mathcal{Y}} \psi(y) d\nu(y) \right\} \tag{2}$$

- We prove that the OT map T^* is the solution of the inner maximization problem (2), see **Lemma 4.2** in our paper.

Unequal Dimensions of Input and Output

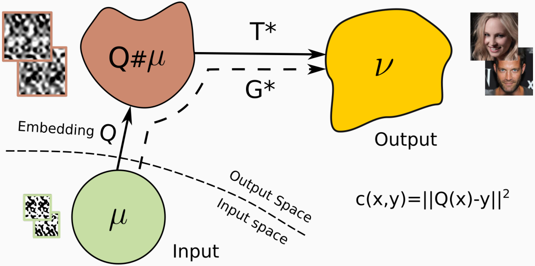


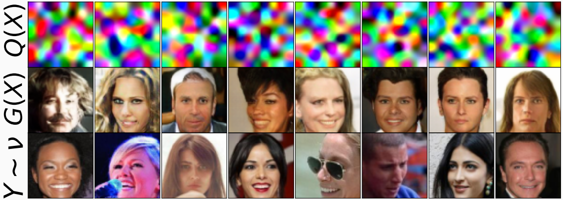
Figure 4: The scheme of our approach.

$$\inf_{\psi} \sup_{G} \left\{ \int_{\mathcal{X}} \{ \langle Q(x), G(x) \rangle - \psi\big(G(x)\big) \} \, d\mu(x) + \int_{\mathcal{Y}} \psi(y) d\nu(y) \right\} \tag{3}$$

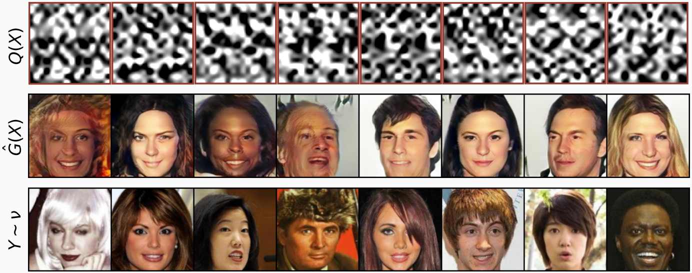
- We prove that the OT map G^* and potential ψ^* are the solutions of the saddle point problem (3).

Overview

Generative Modeling: Qualitative Results



(a) CelebA, 64×64, RGB, FID: 6.5



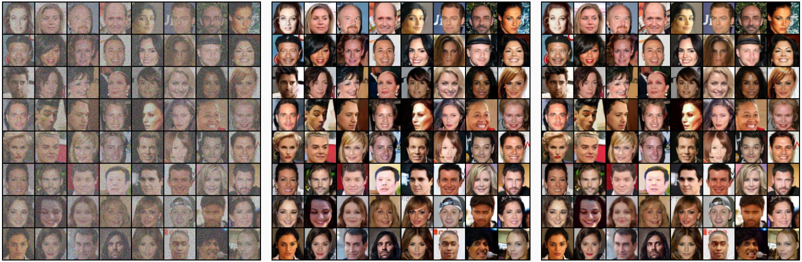
(b) CelebA, 128×128, RGB, FID: 24.58

Generative Modeling: Quantitative Results

Table 1: Results on CelebA, 64x64 dataset.

Model	**Related Work**	**FID** $\downarrow$
DCGAN	Radford et al. (2016)	52.0
DRAGAN	Kodali et al. (2017)	42.3
BEGAN	Berthelot et al. (2017)	38.9
NVAE	Vahdat & Kautz (2020)	13.4
NCP-VAE	Aneja et al. (2021)	5.2
WGAN	Arjovsky et al. (2017)	41.3
WGAN-GP	Gulrajani et al. (2017)	30.0
WGAN-QC	Liu et al. (2019)	12.9
AE-OT	An et al. (2020a)	28.6
W2GN+AE	Korotin et al. (2021)	17.2
AE-OT-GAN	An et al. (2020b)	7.8
OTM	Ours	6.5

Unpaired Restoration



(a) Noisy **(b)** Pushforward **(c)** Original

Model	**Denoising**	**Colorization**	**Inpainting**
Input	166.59	32.12	47.65
WGAN-GP	25.49	7.75	16.51
OTM-GP (ours)	10.95	5.66	9.96
OTM (ours)	5.92	5.65	8.13

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Summary

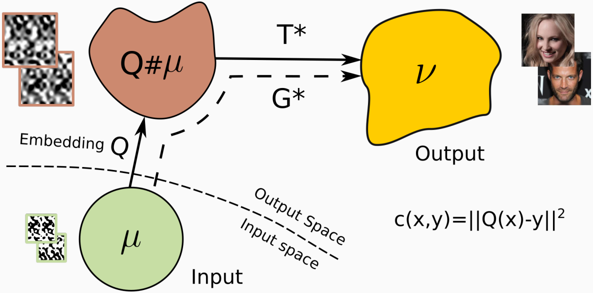
- We proposed to fit the OT map in quadratic transport cost $\mathcal{W}_2^2(\mu, \nu)$ which acts as a generative map.
- We developed an end-to-end solution for equal and unequal dimensions of input and output distributions.
- We demonstrated OTM in unpaired restoration tasks: denoising, colorization, and inpainting.

Conclusion

Generative Modeling with Optimal Transport Maps

Stop by our poster to learn more about our research.

arxiv.org/pdf/2110.02999.pdf



GitHub

github.com/LituRout/OptimalTransportModeling

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