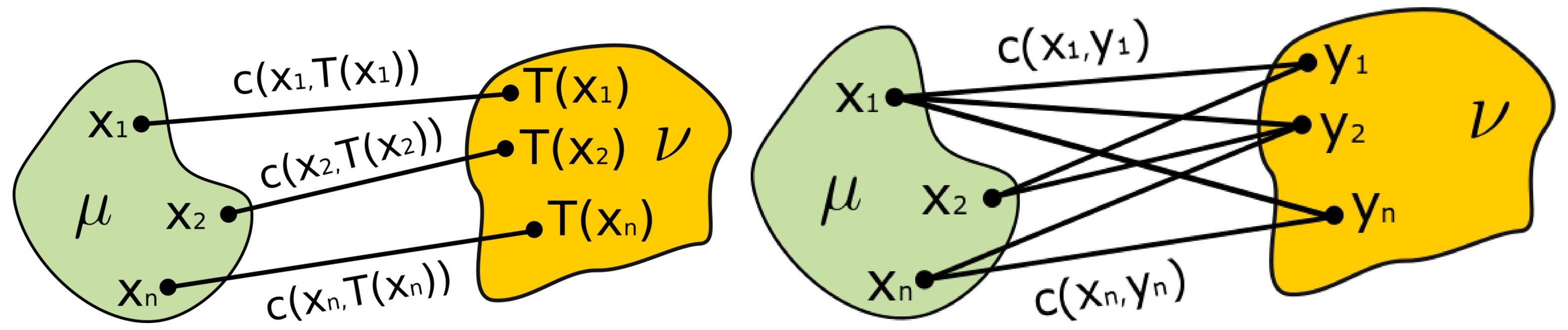
Generative Modeling with Optimal Transport Maps

Litu Rout[1] Alexander Korotin [2] Evgeny Burnaev [2]

[1]Space Applications Centre (SAC), ISRO [2]Skolkovo Institute of Science and Technology

Introduction to Optimal Transport



(a) Monge formulation (1785). (b) Kantorovich formulation (1942).

Monge Problem (MP):

- $\text{Cost}(\mu,\nu) \stackrel{\text{def}}{=} \inf_{T_{\#}\mu=\nu} \int_{\mathcal{X}} c(x,T(x))\,d\mu(x)$.
- The feasible set of solutions can be empty.
- MP does not allow mass splitting.

Kantorovich Problem (KP):

- $\text{Cost}(\mu,\nu) \stackrel{\text{def}}{=} \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\,d\pi(x,y)$.

Kantorovich Dual Problem (KDP):

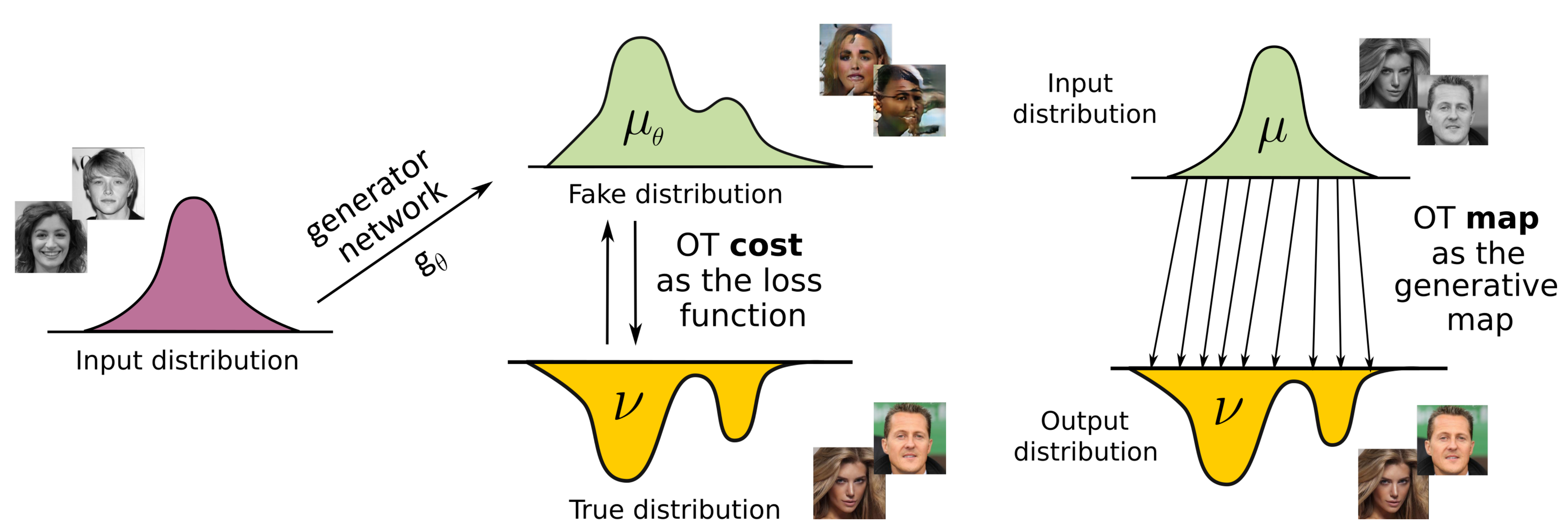
$$\text{Cost}(\mu,\nu) = \sup_{(u,v)} \left\{ \int_{\mathcal{X}} u(x)d\mu(x) + \int_{\mathcal{Y}} v(y)d\nu(y) : u(x)+v(y) \le c(x,y) \right\},$$

Using c-transform, i.e., $v^c(x) = \inf_{y\in\mathcal{Y}} \{c(x,y) - v(y)\}$, KDP becomes (Villani, 2008):

$$\text{Cost}(\mu,\nu) = \sup_{v} \left\{ \int_{\mathcal{X}} v^c(x)d\mu(x) + \int_{\mathcal{Y}} v(y)d\nu(y) \right\}$$

- KDP allows optimization over one functional u or v.

Optimal Transport in Generative Models



(a) OT cost as the loss for the generator. (b) OT map as the generative model.

Optimal Transport as the Generative Map

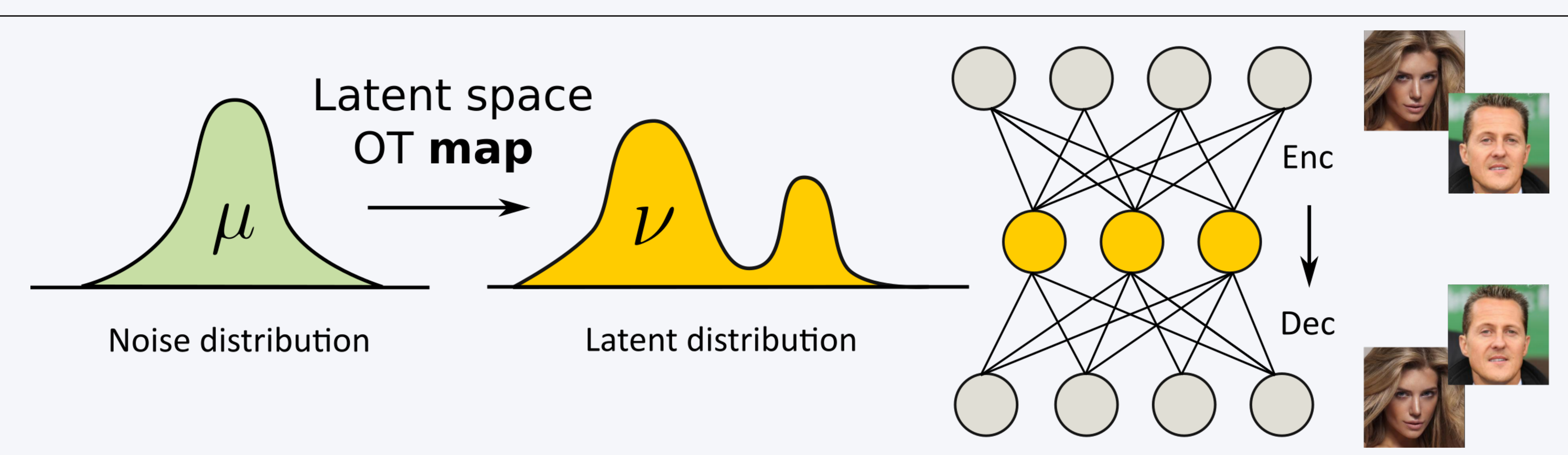


Figure 3. The pipeline of most prevalent approaches.

- Compute OT maps in latent spaces of autoencoders.
- OT maps are not considered in ambient spaces.
- Lack scalability due to poor expressivity of ICNNs.

Proposed Optimal Transport Modeling

Equal Dimensions of Input and Output

With $\psi(y) \stackrel{\text{def}}{=} \frac{1}{2}\|y\|^2 - v(y)$, the c-transform of Kantorovich potential $v(y)$ becomes:

$$\begin{aligned} v^c(x) &= \inf_{y\in\mathbb{R}^D} \left\{ \frac{1}{2}\|x-y\|^2 - v(y) \right\} \\ &= \frac{1}{2}\|x\|^2 - \sup_{y\in\mathbb{R}^D} \{\langle x,y\rangle - \psi(y)\} = \frac{1}{2}\|x\|^2 - \overline{\psi}(x). \end{aligned} \quad (1)$$

Substituting (1), KDP simplifies to

$$\text{Constant}(\mu,\nu) - \inf_{\psi} \left\{ \sup_{T} \int_{\mathcal{X}} \{\langle x,T(x)\rangle - \psi(T(x))\}\,d\mu(x) + \int_{\mathcal{Y}} \psi(y)d\nu(y) \right\} \quad (2)$$

- We prove that the OT map T^* is the solution of the inner maximization problem (2), see **Lemma 4.2** in our paper.

Proposed Optimal Transport Modeling

Unequal Dimensions of Input and Output

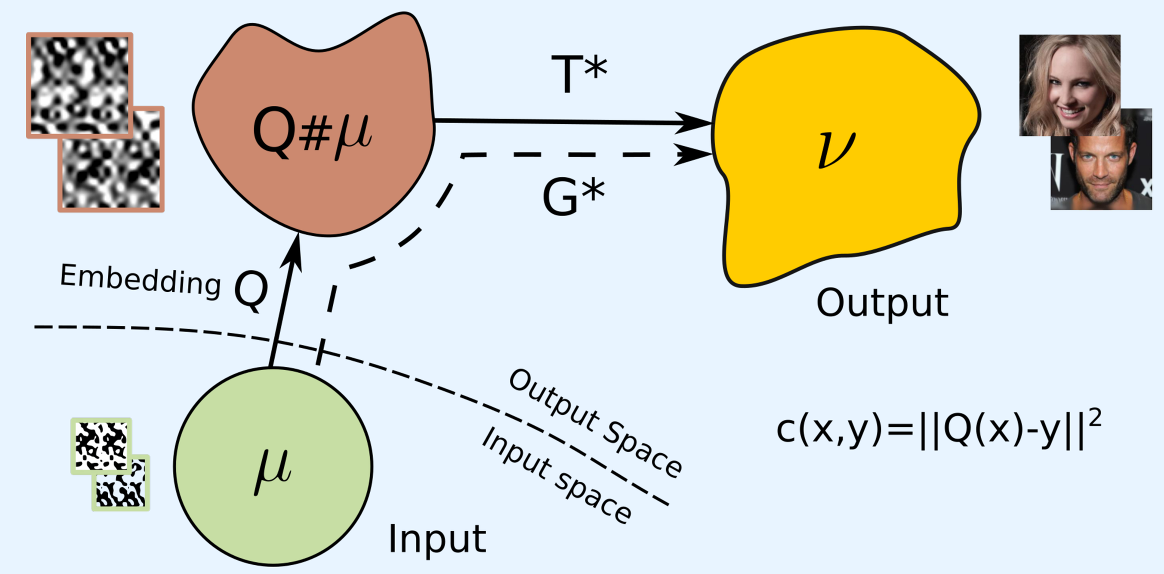


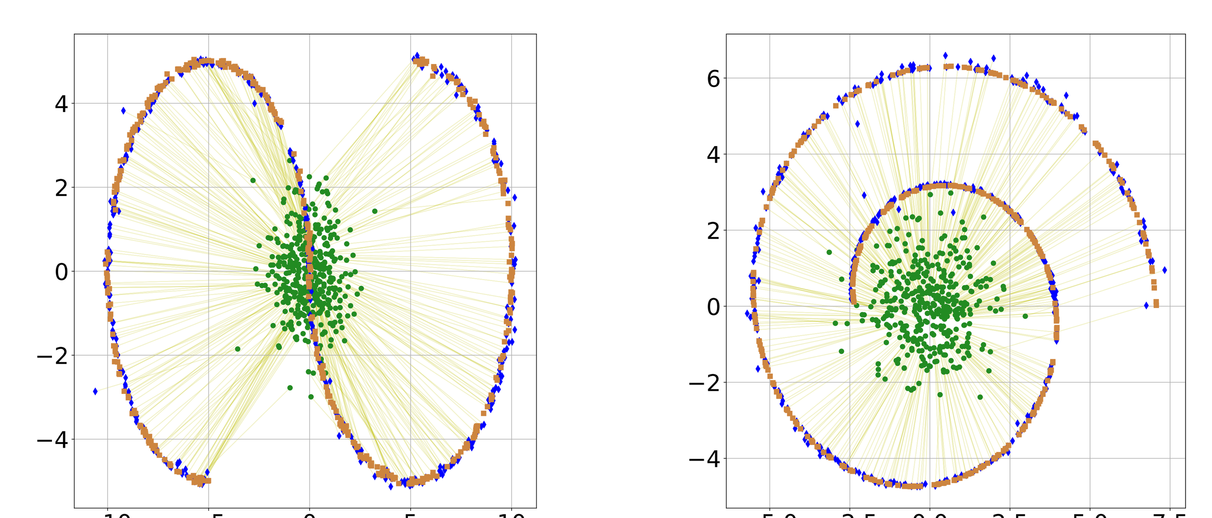
Figure 4. The scheme of our approach.

$$\inf_{\psi} \sup_{G} \left\{ \int_{\mathcal{X}} \{\langle Q(x),G(x)\rangle - \psi(G(x))\}\,d\mu(x) + \int_{\mathcal{Y}} \psi(y)d\nu(y) \right\} \quad (3)$$

- We prove that the OT map G^* and potential ψ^* are the solutions of the problem (3).

Experimental Results

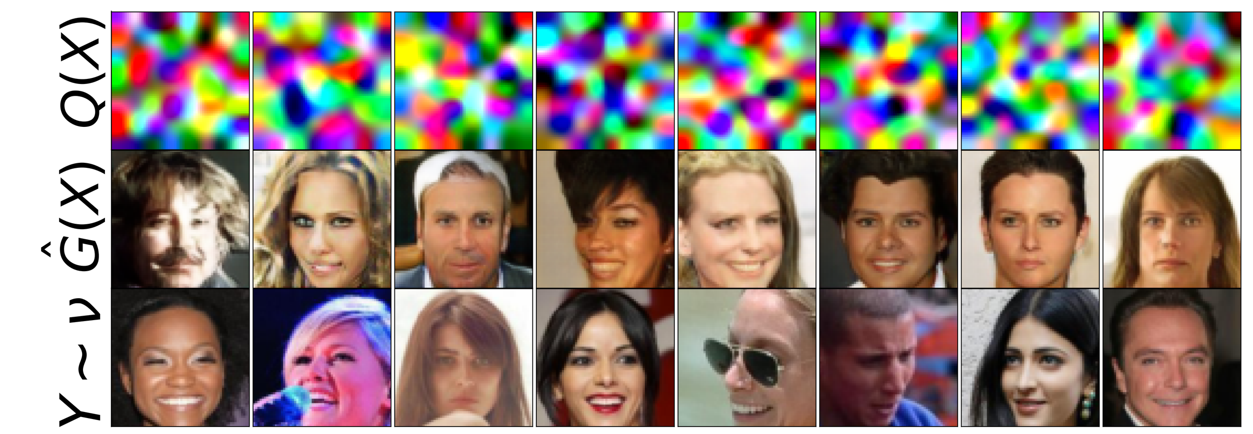
Generative Modeling: Toy Dataset



(a) S Curve (b) Swiss Roll

Experimental Results

Generative Modeling: Qualitative Results



(a) CelebA, 64 × 64, RGB, FID: 6.5

Experimental Results

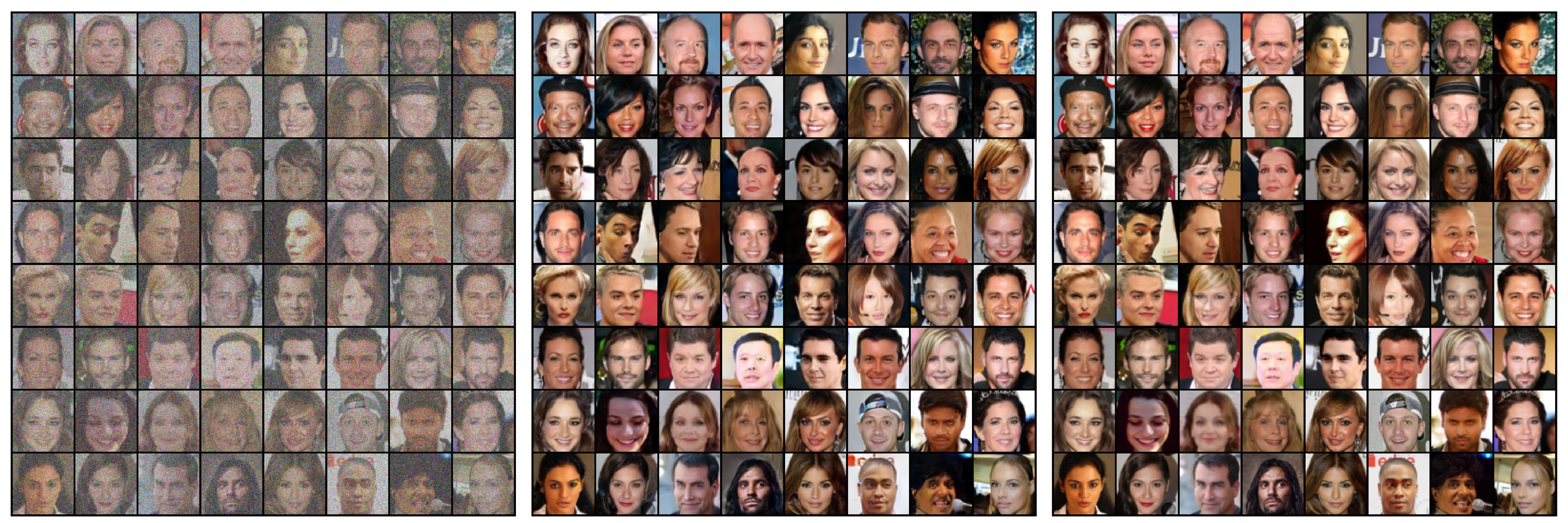
Generative Modeling: Quantitative Results

Table 1. Results on CelebA, 64x64 dataset.

Model	Related Work	FID ↓
DCGAN	Radford et al. (2016)	52.0
DRAGAN	Kodali et al. (2017)	42.3
BEGAN	Berthelot et al. (2017)	38.9
NVAE	Vahdat et al. (2020)	13.4
NCP-VAE	Aneja et al. (2020)	5.2
WGAN	Arjovsky et al. (2017)	41.3
WGAN-GP	Gulrajani et al.(2017)	30.0
WGAN-QC	Liu et al. (2019)	12.9
AE-OT	An et al. (2020)	28.6
W2GN+AE	Korotin et al. (2021)	17.2
AE-OT-GAN	An et al. (2020)	7.8
OTM	Ours	6.5

Experimental Results

Unpaired Restoration



(a) Noisy (b) Pushforward (c) Original

Model	Denoising	Colorization	Inpainting
Input	166.59	32.12	47.65
WGAN-GP	25.49	7.75	16.51
OTM-GP (ours)	10.95	5.66	9.96
OTM (ours)	5.92	5.65	8.13

Project Page: `github.com/LituRout/OptimalTransportModeling`